Preface To 2017 Edition

This book is the beginning of a third attempt by the publisher to join a quiet revolution in academic publishing. And the timing of this revolution couldn't be better.

The quiet revolution I'm talking about is the burgeoning world of online self-publishing-in particular, the self-publishing of academic textbooks. There was a time, not so long ago, when publishing a mathematics textbook required a deal with one of the major publishers-and unless you were a Fields Medal winner at Harvard, you were basically at their mercy even if they decided to bless you with a contract for your text. They dictated length, what they wanted in the book, how many graphics, that section's too tough for American students, complete ownership of the copyright for a century if the book was successful-on and on and on. And after all that pain and suffering, the resulting text is so expensive, the author can't even recommend it to his or her own students because they'll have to take out a loan to purchase it. This is why until recently so many practicing scientists-and mathematicians in particular-would cringe when you suggested writing a textbook. Even if writing such a book in their field was a labor of love for them, the misery of getting it published sucked all the joy out of it.

Sadly, this is part of a larger social regression over the last 30 years in America of making higher education a career necessity that simultaneously, is purely a profit making machine with increasingly restricted access to the children of the wealthy. Horrifyingly, with the charter school/academic reform movement, this thinking is even now spreading like an aggressive, hedge fund fueled cancer, to grade and secondary schools where quality public schools are rapidly becoming extinct through intentional neglect. The result is an entire generation that's crippled with lifelong debt and steadily diminishing earning prospects, while the prospects of their children even being able to read, let alone go to college, erodes with each new administration. And this is a crisis on both political sides borne of the increasingly plutocratic nature of our nation.

(I was truly hoping to keep this preface entirely apolitical. But I realized it's really impossible to coherently explain the motivations for this project without *some* digression on the current situation in American society.)

As someone who was trained at The City University of New York, this situation makes me want to put my fist through a wall. An all-too-possible future nightmare I wake up consistently with now My future descendants' in the 23rd century subsistence farming to survive while wondering what magic holds the moon up, if it's really made of cheese and instinctively kneeling in nearly religious terror to passing Lords and Ladies of America. Meanwhile, those practically immortal, limitlessly wealthy and bored aristocrats snicker and make plans to kidnap and rape their daughter as after-dinner entertainment and a moment's respite from their tedium. No one may be able to prevent this future. Indeed, it may be a predetermined outcome of American history and culture. But we shouldn't stand silently by and let it happen. In particular, education should be a right every citizen should have a realistic chance at and if our government is being paid off to ensure only the children of the privileged have that right, then it's up to us to create alternatives.

I decided that I was going to do what little I could to ensure that future didn't come to pass. And one of the best ways to do that was to create methods of self-education available to all people via the web and low cost purchases. Since my area of interest is mathematics and the physical sciences, I'd focus on that.

My first, rather shallow attempt was when, as a graduate student, I began the blog *Tables,Chairs And Beermugs* (https://tableschairsandbeermugsmathemagician.blogspot.com)-the title being an ode to Hilbert's famous quote and referring to the rather ubiquitous subject matter of the blog. Despite the planned diverse nature of the blog's content, my health problems and distractions of my personal life prevented me from making the blog as successful as it could have been. Part of the problem was lack of focus in the posts. I'm now turning back to it and focusing its' subject matter on purely mathematical matters: Textbook reviews, academic commentary and research topics. I hope this will not only improve the blog's quality, but allow it to attract more of an audience on the virtual data torrents. But even so, it wouldn't make a dent in my stated goals.

The second, much more significant attempt by me to make a contribution to this revolution was the massive Wordpress website, *Tuloomath* (www.tuloomath.com), to which I dedicated a year and a half of my precious life with no formal computer science training building. The website's purpose is clear in the name, which is an acronym for the following mouthful: **The Universal Lyceum Of Online Mathematics.** A lyceum is a library or repository of information. That's precisely what this was intended to be-a complete one stop link collection for quality free sources of study-lecture notes and online textbook drafts-on the internet for impoverished students of mathematics at all levels from secondary school algebra and geometry to university PhD research topics. Sadly, the response, while generally positive from users, hasn't gotten the traffic I'd desired. Obviously, to be effective, such a site needs to be constantly revised as nothing is more ephemeral then links to posted lecture notes on the World Wide Web. But even if this website becomes a beating heart the web academic body, it's sadly only partially a solution to the inability of university students to buy more than a handful of quality mathematics textbooks.

The publication of the text you're holding in your hand is the beginning of the major third stage of my contribution to the revolution.

This book, produced by myself with permission from Houghton Mifflin Harcourt and published via Createspace in both inexpensive paperback and e-book format, is the first title of a new publishing company whose name I'll keep under my hat until it's been formalized. The goal of this company will be to produce a line of inexpensive quality mathematics textbooks that are affordable for classroom use by both students and teachers alike worldwide, as well as to encourage independent study. I plan to publish both original works and reprints of out of print classic textbooks.

The text you hold in your hand is of the latter breed.

This book is a reprint of a classic mathematics textbook by the famous French mathematician, Henri Cartan. If you're a student reading this and don't know who the hell Henri Cartan is, that's ok. That's why I put the About The Author section on the last page (check it out, please). If you're willing-as I hope you are-to find out more, especially after using this fine textbook, you can Google his name. And I hope in doing so, you'll learn a bit of history and find the context it provides for the origins of the ideas of the language of mathematics-of all sciences, really-intriguing. When you understand what questions lead to the formulation of specific concepts-and more importantly, the people that came up with them-they come to life in your understanding a manner merely studying concepts never do.

Some publishing history: The book began life as the first semester of a year-long undergraduate analysis course given by Cartan in 1965/66, at the Faculty of Sciences in The University of Paris. The original French edition was published as *Calcul differential* by Hermann in 1967. An English translation was then prepared in 1971 by John Moore (then of Princeton University) and Dale Husemoller (then of Haverford College) and published by Kershaw Publishing Company in 1971 with the rather deceptive title, *Differential Calculus*.

I've somewhat brazenly retitled the book for this edition. Why did I do that when it was known by the title *Differential Calculus* in each of the languages it's seen print in by such a distinguished author?

Well, to be honest, in 2017, the original title is quite misleading to modern audiences-although when it was written, it was a propos for its intention at the time.

If an American college student today were to pick up a book with that title, they'd be expecting a standard calculus book that just covers plug and chug methods of differentiation of various functions, derivative tests and graphing and maybe some differential equations-all peppered with lots of pictures and applications. Maybe a couple of proofs of theorems that applied students could skip.

When they opened Cartan's book-they'd get one hell of a surprise. And from their perspective, not a good one. Some backstory on that surprise, the development of Cartan's famous book and its sequel (more on that later) would be appropriate here.

Back in the halcyon days of the mid-1960's when Cartan was giving his original lectures in Paris, mathematics courses were taught quite differently at top universities then how they're taught today in both America and Europe. It was much more rigorous and abstract then it is today. Of course, incoming freshmen were far stronger in their secondary school mathematical and scientific preparation then they are today. In America, at the height of the space program, it was fairly common for the usual "plug and chug" calculus to be taught in high school to most students planning on entering university regardless of major. Inequalities, simple ε-δ limit proofs and difficult integration computations with hypergeometric and inverse trigonometric substitutions were all a standard part of university freshman calculus courses. Indeed, it wasn't uncommon for strong high-school juniors or seniors in math or physics to be simultaneously enrolled in local university undergraduate mathematics and physics courses! (Today in America, one finds that an extreme rarity that occurs only with the country's most gifted students destined for the very best universities, such as Harvard or Yale.)

In France at the height of the influence on math education of the Bourbaki group, this purity zealotry in mathematical training was taken to an absurd apex. Cartan himself was one of the main architects of the educational curricula machine that propelled the French students to such dizzying heights in the 2 decades following World War II-although he had been experimenting teaching such “modern” classes all his career. In many ways, the completely modern mathematical education one would find in France in the 1960’s was the culmination of over 30 years of Cartan and his fellow Bourbaki’s influence on students and in turn, the French education system. This “Bourbaki takeover”, which produced the book you hold in your hands as well as its sequel,was summed up exquisitely in the following reminisces of Pierre Cartier of his student days:

French mathematics was at a turning point. The undergraduate curriculum (even in its enhanced form for the "classes preparatoires") (i.e. advanced courses designed for students pursuing postgraduate work in mathematics or the sciences) *was a mix of coordinate geometry, synthetic geometry (based on the "theorem" that every one-to-one correspondence between the points of a projective line is given by a Mobius transformation),differential calculus with applications to geometry and kinematics. The foundations were sometimes shaky, there was hardly any hint of groups of transformations (in geometry), and the use of matrices was ignored or not advised. The good teachers dared to give the foundations of the real number system, but in the absence of set-theoretical terminology, the exposition was quite obscure. In the land of Lebesgue, hardly any mention was made of the Lebesgue integral, and we had to learn Lie groups from the thesis of Elie Cartan or in Pontrjagin* (i.e. L.E. Pontrjagin’s classic *Topological Groups*, published in 1932) *! By a sequence of well-planned steps, Cartan made General Bourbaki win! He managed to hire the ambitious youth at La Sorbonne: Schwartz, Choquet, Dixmier, Godement, and Chevalley. In 1957 the takeover was complete (except for Andre Weil, who was never forgiven for his refusal to be drafted in 1939, at the beginning of WWII!). The curriculum was deeply renovated, and the textbook of textbooks became Bourbaki (whose golden age extends from 1950 to 1975!). The forceful gesticulations of Dieudonne, as well as the power of persuasion of Choquet and Lichnerowicz (both not members of Bourbaki) convinced everyone to worship general topology, linear algebra, functional analysis, and group theory.* (Cartier, Pierre, “A Tribute to Henri Cartan”, *Notices of The AMS*, **57** *(8)*, 2010, pg 958)

A wonderful anecdote attributed to the famous Russian mathematical physicist Vladimir Arnold encapsulates the atmosphere at the time: Arnold came to visit Paris in the early 1970’s and was visiting a local grade school. The teacher introduced Arnold to a student and said he was the classes’ best at mathematics. He smiled and asked the student,” Young man, tell me, what’s 5 times 3?” The child asserted strongly,”3 times 5 because addition is commutative!” The child’s retort was symbolic of general French mathematical education in the Bourbakian tradition. To which the Russian scientist-trained in the renowned Moscow State mathematical tradition where application and theory went hand in hand-shook his head in disbelief.

Mathematics courses at all levels were taught at French schools with ultimate rigor and generality. Grade school students were taught “the new math”, brimming with set theory, basic logic, group theory and matrix computations alongside addition and multiplication tables. The high school geometry courses that resulted had all the pictures removed and resembled a linear algebra course where all Euclidean geometry was developed via groups of linear transformations of the plane. In high school (!) calculus courses, applications were downplayed and careful proofs were given of most important results via ε-δ limit and convergence arguments. So it's really not surprising at top French universities at the height of the Bourbaki era, first year calculus was taught on normed vector or metric spaces where the derivative was a linear approximation to operators defined on Banach spaces. In other words, a freshman university calculus course in France then was what today in the U.K. or Germany would be called a basic analysis course and what we would call in America today an intermediate level "Analysis 2" course. (It is interesting to note that this model of mathematical education still exists today in a number of European countries, such as the U.K., Germany and the Netherlands-albeit in somewhat weakened form. But mathematical education in these countries is still **far** superior to what's currently available to the average student in America.)

Cartan's goal with this course-and the 2 texts that resulted from it-was to present a Bourbakian version of differential and integral calculus for beginning students. Assuming such incoming freshman would have the complete command of basic modern mathematical language and rigor formerly expected of an advanced undergraduate mathematics major, these students would be expected to learn calculus in completely modern language and at the highest level of abstraction. In other words, these amateurs would be forced to learn calculus using the language of modern research mathematicians. It would be presented with maximal generality on Banach spaces as a study of linear transformations between defined subspaces. While Cartan didn't eschew the use of visual aids and physical applications completely as some of the hardcore zealots of the era did, he certainly made certain the students didn't confuse a graph or a mechanics calculation for a definition or a proof of a theorem! (It's really difficult to imagine most teenage freshman fresh out of high school-even talented ones-being able to handle such a course effectively. I'd love to know what the passing rate was back then, seriously…….)

Today's calculus courses in America are of course, **very** different. The majority of prospective students have much weaker backgrounds with the gradual and pronounced degeneration of our schools at every level. The resulting courses are in general far more applied and far less theoretical so they can be pitched at the lowest common denominator for maximum profit rather than educational value. A strong and diverse education for students-particularly in America- is no longer a top priority. (In fact, quite the contrary. In America, many of our current leaders seem to think unless you can afford treatment, **you should be allowed to die coughing up blood if you have cancer.**) It's hard to imagine Cartan's course being used to teach students calculus except in honors courses at the very strongest programs such as Harvard or The University of Chicago. And if it **was** used for such a course, it would probably have to be extensively supplemented with material from a standard calculus text.

And to be honest, even if they had the proper skill and background, I'm not sure we'd want them to. The multitude of applications and geometric intuition behind calculus is just as important a part of its mastery as the rigorous construction of the real numbers and topological properties. While the calculation aspect of calculus isn't completely omitted from Cartan's text, particularly in the chapter on differential equations, it is certainly downplayed greatly. I don't think most mathematicians and mathematical educators would consider this a good way to train even the strongest beginners in calculus.

However, considering Cartan as a text for the next step of their training-as an intermediate course in classical analysis on abstract spaces is-as the late, great Art Rust Jr. used to say when I was a kid-a horse of another garage.

I believe Cartan's text(s), because of their unusual structure and choice of topics, can be used in a number of different kinds of courses. First and foremost, for students with the proper background-particularly those who plan to later enter graduate school in mathematics-Cartan's text(s) would make for a remarkable course in classical analysis at the junior-senior level. The focused, selective nature of these texts would help analysis students improve their understanding of the specific machinery of calculus in a manner that sometimes gets lost in The Big Picture of learning analysis. The geometric bent of Cartan's books will also help prepare them for later courses in differential geometry and advanced topology. A second, perhaps even more valuable use of these books would be as very versatile supplements. The inexpensive price of the book(s) would make them provide a more visual perspective then standard analysis texts such as Rudin's classic (12) or even better, Hoffman's much more compatible and equally inexpensive classic (9) usually provide. It could also act in a similar manner as guided study to supplement an honors calculus course with a student of good linear algebra background. A third possible use of these books could be for seminars or reading courses for math majors on topics that are usually presented at too sophisticated a level for such students. Since each chapter of each book stands alone with the proper background, the books can be used selectively. For example, the second part of *Calculus* could be used to supplement a standard text in differential equations in an advanced undergraduate differential equations course to provide theoretical background on solution spaces and existence theorems at a pre-graduate level.

It only made sense to me that before reissuing the book, it should be retitled to better reflect its actual content and intent. I've made **very** minimal changes to this book. In fact, the bulk of this edition is an unaltered reprint of the original 1971 Kershaw edition. When serving prime rib, one should cook it to medium rare and let the quality of the ingredients speak for itself in the finished dish. Therefore, it should be served with as few accoutrements as possible and only those that are judged necessary. The new title, this preface and the supplemental references are really all the new additions there are. Consider them horseradish and good rustic bread to be served only to maximize the dining experience of the main course.

I experimented with quite a few titles before settling for the one that graces the cover of this edition: *Differential Calculus On Normed Spaces: An Analysis Course.* Indeed, there was briefly an e-book version offered for sale on Amazon with the very different and unwieldy title, *A Modern Introduction To Classical Analysis: Derivatives and Differential Equations* as well as a slightly different preface and bibliography. I was too hasty in publishing, a lesson I'll take to heart for the future. I hope these minor but important changes improve the utility of the book for students and teachers. (For those who ordered the e-book with the original title and content-it may be a collector's item when I'm famous, so hold onto it!)

It's always important when learning a new subject in mathematics, especially one as important as undergraduate analysis, to see as many different perspectives as possible-which is why the cost of textbooks is so distressing. It's one of the reasons I've worked hard to make Cartan's book available again as it and its' sequel together (more on that later) gives one of the best concise presentations of intermediate analysis I've ever seen. It's also a wonderfully unorthodox presentation. Most standard texts on analysis use metric spaces rather than Banach spaces like Cartan. Both approaches have their pros and cons for presenting intermediate analysis on abstract spaces. The main advantage of metric spaces is they provide a simple model of abstract topological spaces by straightforward generalization of the Euclidean distance function. They also require only naïve set theory to build most of the machinery of limits and continuity of functions. Ideally, when one is presenting classical analysis on abstract spaces, one would like the topics to mirror those in a calculus course, merely presented at a much higher level. However, since many aspects of classical analysis rely heavily on linear mappings-particularly abstract differential calculus and the generalizations of the Fundamental Theorem of Calculus-using an approach solely on abstract metric spaces becomes quite problematic. Usually, the derivative has to either be defined in Euclidean or normed spaces or some kind of vector space structure needs to be artificially constructed on the abstract space i.e. the tangent space on a differentiable manifold. Even complete metric spaces cannot really support an abstract notion of derivative (and therefore differential equations) without an imposed vector space structure of some kind.

When one uses normed spaces, particularly Banach spaces, this problem disappears. A Banach space B is the infinite dimensional generalization of R^n . As such, it possesses essentially the same algebraic structure as R^n as a vector space over the fields $\mathbb{R}$ or $\mathbb{C}$ and if given a norm, it generates a metric topology on B. If the norm put on B is the ordinary Euclidean distance norm, then the topology generated on B is the standard topology on R^n .(Or, equivalently, the product topology of R^n .) More importantly, the completeness of B (i.e. every Cauchy sequence converges with respect to the norm) gives it essentially the same topological properties as R^n with the same norm. The end result is that the theory of differential and integral calculus of R^n can be directly generalized to Banach spaces and for the purposes of calculus, it's irrelevant whether or not B is finite or infinite dimensional. When we begin considering more general kinds of structures on Banach spaces-such as function spaces-this is no longer true. But this brings us to the doorstep of functional analysis and for our purposes, it would be wise to close this door at this point.

Another advantage of this approach is that it unifies both single and multivariable analysis for both the differential and integral calculus on Banach spaces in a "coordinate free" manner- a single formulation covers all special cases. For example, the usual development of the derivatives on the real line and the more general n-dimensional Jacobian derivative are both special cases of the Frechet derivative. The usual machinery of multivariable calculus-gradients, directional derivatives, partial derivatives-only have to be introduced in the case of specific calculations and examples defined in terms of local isometries of open sets in the ambient space. Lastly, the emphasis on vector space properties allows for a geometric approach to typical topics in undergraduate analysis via local isometries of Banach spaces. The most important and beautiful "local" computations and examples that appear are given for differential equations, where the theory of contour curves are images of isomorphisms of the solution subspaces.

Let's now briefly describe the book's contents somewhat.

The book is divided into two chapters. The first develops the abstract differential calculus. The introductory section provides an overview of the algebra and topology of Banach spaces, including norms, metrics, completeness, limits, convergence, isomorphisms and dual spaces along with important examples. An introduction to multilinear algebra is given via the exterior product and a brief digression into Banach algebras. Then the Frechet derivative is defined and proofs are given of the two basic theorems of differential calculus: The mean value theorem and the inverse function theorem. The chapter proceeds with the algebra of polynomials in Banach spaces,the corresponding study of higher order derivatives and a proof of Taylor's formula. It closes with a study of local maxima and minima including both necessary and sufficient conditions for the existence of such minima.

The second chapter is devoted to differential equations and really is the strength of the book. Indeed, it can easily be used for an honors course on differential equations that gives a completely rigorous treatment at the pre-functional analysis level. To my knowledge, there are very few such treatments for students. The general existence and uniqueness theorems for ordinary differential equations on Banach spaces are proved and applications of this material to linear equations and to obtaining various properties of solutions of differential equations are then given. Finally the relation between partial differential equations of the first order and ordinary differential equations is discussed and developed in detail.

The prerequisites for this book are

a) A rigorous first course in calculus using naïve set theory, the ε-δ definitions of convergence and limits, precise formulations of derivatives, integrals, and sequences and series on the real line (or equivalently, a course in one variable advanced calculus or elementary analysis). The basic definitions of topology (metric and topological spaces, open and closed sets, etc.) will be needed as well.

b) A careful course in linear algebra on abstract vectors spaces with linear transformations as well as fluency with matrix computations and arbitrary bases.

and

c) A basic course in differential equations with the usual solution techniques is advised as well. A knowledge of the computational aspects of multivariable calculus will also be needed for some parts of the book. Any standard calculus book will do for this.

(Since one the one of this book's purposes is to provide a low-cost textbook for students, I'll try to provide references that are equally inexpensive whenever possible.)

I want to add in closing another reason I've gone through the trouble of republishing this particular book. The second half of the course was also published by Hermann as *Formes Differentielles* in 1967. It was also then translated by the Moore/Husemoller duo and also published by Kershaw as *Differential Forms* in 1971. In 2006, textbook on the cheap titan Dover Books republished the second half translation-but for some inexplicable reason, did not republish the first half. This has always baffled me as a student since as far as I know, the first half hadn't been republished anywhere else and there were no plans to do so at the time. This has become a real headache for students looking to use the second half to learn differential forms Most books that cover the needed prerequisites of differential calculus on Banach spaces for the second volume are either much more difficult than Cartan's first volume-for example, the text (10) as well as functional analysis texts-or don't cover the proper selection of material for the second volume to make sense. Unless as student had access to a comprehensive academic library that actually had a copy of the first volume, using the Dover edition of the second part as anything but a supplementary text was extremely problematic. Since the second volume is just as wonderfully clear unorthodox and readable as the first, what was really needed was someone to put the first volume back into print in an inexpensive edition so students and teachers could buy both volumes and pair them for a one year complete analysis course on Banach spaces. So that's what I've done, finally reuniting the 2 long-lost brethren for students and I hope now many will choose to study Cartan's beautiful course as it was originally presented and intended.

For all the teachers and students of analysis who are about to become students of Cartan from beyond the grave through these wonderful and unique texts, I hope you will find this book (and its' sequel!) as enlightening and joyful as Cartan's students did. I wish to thank Mary Rodriguez at Houghton Mifflin Harcourt for granting me permission to make this classic text available again for a new generation of students. Here's hoping that it acts as the launching pad for an entire line of inexpensive sources that open the door to the wonders of mathematics for all students regardless of background, class or creed.

Enjoy!

Karo Maestro aka The Mathemagician

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